

An Analysis of Harmonic Distortion and Integral Nonlinearity in Digital to Analog Converters

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□ **Abstract** - A Fourier technique is developed that relates a digital to analog converter's (DAC's) characteristic to its output spectrum and a DAC's integral nonlinearity to its distortion. Therefore, a DAC's spurious free dynamic range, signal to distortion ratio and signal to noise plus distortion ratio can be calculated from its characteristic.

I. INTRODUCTION

A linear digital to analog converter (DAC) transforms a B bit digital signal, $x(n)$, into an analog output signal, $y_l(nT)$, such that $y_l(nT) = qx(n)$ where $x(n)$ is the DAC's digital input signal, T is the DAC's sampling period, q is a constant that represents the DAC's quantization step sizes or code widths, and n is an integer that indexes the sequence x . Because of variations in circuit fabrication processes, DACs contain mismatched circuit components that cause the code widths to be functions of $x(n)$ and not constants. Variable code widths that are a function of $x(n)$ add a nonlinear transformation, referred to as integral nonlinearity (INL), to the DAC's linear transformation. Nonzero INL generates harmonic distortion in a DAC's output. This harmonic distortion reduces a DAC's spurious free dynamic range (SFDR), signal to distortion ratio (SDR), and signal to noise plus distortion ratio (SNDR).

In this paper, a Fourier analysis technique is developed that relates a DAC's transfer characteristic to its frequency spectrum. This technique can also be used to relate a DAC's INL to its harmonic distortion. Using these analyses, a DAC's SFDR, SDR and SNDR can be calculated from its transfer characteristic.

II. FOURIER SERIES REPRESENTATION OF A DAC'S OUTPUT

If the transformation \mathcal{T} represents a B bit DAC's transfer characteristic, the DAC's output, $y(nT)$, can be written as

$$y(nT) = \mathcal{T}[x(n)] \quad (1)$$

where \mathcal{T} is a nonlinear transformation that can be described as

$$\begin{aligned} \mathcal{T} = & \left\{ \mathcal{T}[x(n)] = \bar{q}x(n) + INL_{x(n)-x_0} : x_0 \leq x(n) \leq x_0 + 2^B - 1 \right\} \\ = & \left\{ \bar{q}x_0 + INL_0, \bar{q}(x_0 + 1) + INL_1, \dots, \bar{q}(x_0 + 2^B - 1) + INL_{2^B - 1} \right\}, \end{aligned}$$

\bar{q} is the DAC's average code width, $\bar{q}x_0$ is the DAC's zero input offset, $x(n)$ is the DAC's digital input sequence that has been amplitude shifted¹ so that its minimum value equals x_0 , and INL_{k-x_0} is the DAC's INL at $x(n) = k$. This last statement implies that $INL_0 = 0$. The DAC's output, $y(nT)$, in (1) can also be written as

$$y(nT) = \mathcal{T}[x(n)] = \underbrace{\mathcal{T}_l[x(n)]}_{y_l(nT)} + \underbrace{\mathcal{T}_e[x(n)]}_{y_e(nT)}, \quad (2)$$

where the linear transformation, \mathcal{T}_l , is

$$\begin{aligned} \mathcal{T}_l = & \left\{ \mathcal{T}_l[x(n)] = \bar{q}x(n) : x_0 \leq x(n) \leq x_0 + 2^B - 1 \right\} \\ = & \left\{ \bar{q}x_0, \bar{q}(x_0 + 1), \dots, \bar{q}(x_0 + 2^B - 1) \right\}, \end{aligned}$$

and the nonlinear transformation, \mathcal{T}_e , that maps $x(n)$ to the output's conversion errors, $y_e(nT)$, is

$$\begin{aligned} \mathcal{T}_e = & \left\{ \mathcal{T}_e[x(n)] = INL_{x(n)-x_0} : x_0 \leq x(n) \leq x_0 + 2^B - 1 \right\} \\ = & \left\{ 0, INL_1, INL_2, \dots, INL_{2^B - 1} \right\}. \end{aligned}$$

For a B bit DAC, each of the three transformations, \mathcal{T} , \mathcal{T}_l , and \mathcal{T}_e , maps 2^B inputs to 2^B outputs, and therefore each of the transformations can be represented by a finite length sequence. Because each of these transformations can be represented by a finite length sequence, they can be represented by a discrete Fourier series (DFS). In particular, the transformation, \mathcal{T} , can be written as

1. The DAC's digital input sequence has been amplitude shifted so that the DC power of the DAC's digital input signal, $x(n)$, can be made equal to the DC power of the DAC's analog output signal, $y(nT)$.

$$y(nT) = \mathcal{T}[x(n)] = \frac{1}{2^B} \sum_{l=0}^{2^B-1} A(l) e^{j \frac{2\pi}{2^B} l [x(n) - x_0]} \quad (3)$$

where

$$A(l) = \sum_{z(n)=0}^{2^B-1} \mathcal{T}[z(n) + x_0] e^{-j \frac{2\pi}{2^B} l z(n)}$$

If the signal $x(n)$ in (3) is a finite length sequence of length N , then each of the mutually orthogonal sequences, $\exp(j2\pi l [x(n) - x_0] / 2^B)$, is a finite length sequence of length N , and can be expressed as the linear combination of a complete set of mutually orthogonal sequences. Similarly, if the signal, $x(n)$, is periodic with period N , then each of the mutually orthogonal sequences, $\exp(j2\pi l [x(n) - x_0] / 2^B)$, is periodic with period N , and can be expressed as the linear combination of a complete set of mutually orthogonal periodic sequences that have a period N . Therefore, if $x(n)$ is a finite length sequence of length N or a periodic sequence with period N ,

$$e^{j \frac{2\pi}{2^B} l [x(n) - x_0]} = \frac{1}{N} \sum_{n=0}^{N-1} C(k, l) e^{j \frac{2\pi}{N} kn} \quad (4)$$

for $0 \leq l \leq 2^B - 1$, where

$$C(k, l) = \frac{1}{2^B} \sum_{n=0}^{N-1} e^{j \frac{2\pi}{2^B} l [x(n) - x_0]} e^{-j \frac{2\pi}{N} kn} \quad (5)$$

Substituting (5) into (4) and the result into (3),

$$y(nT) = \mathcal{T}[x(n)] = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j \frac{2\pi}{N} kn}$$

where

$$Y(k) = \frac{1}{2^B} \sum_{l=0}^{2^B-1} \sum_{n=0}^{N-1} e^{j \frac{2\pi}{2^B} l [x(n) - x_0]} e^{-j \frac{2\pi}{N} kn} \times \sum_{z(m)=0}^{2^B-1} \mathcal{T}[z(m) + x_0] e^{-j \frac{2\pi}{2^B} l z(m)}$$

By defining the column vector, \mathbf{Y} , such that $Y(k)$ is the k th element of \mathbf{Y} , the Fourier coefficients in (6) can also be calculated using

$$\mathbf{Y} = \mathbf{C}\mathbf{A} \quad (7)$$

where $C(k, l)$ is the element in the k th row and l th column in the matrix \mathbf{C} , and $A(l)$ is the l th element in the column vector \mathbf{A} . The matrix \mathbf{C} in (7) can also be calculated by

$$\mathbf{C} = \left[\text{DFS}_N \left\{ \frac{1}{2^B} \right\} \mid \text{DFS}_N \left\{ \frac{1}{2^B} e^{j \frac{2\pi}{2^B} l [x(n) - x_0]} \right\} \mid \dots \right]$$

where $\text{DFS}_N\{z(m)\}$ generates a column vector containing the N point discrete Fourier series (DFS) coefficients of the sequence $z(m)$. Likewise, the vector \mathbf{A} in (7) can be calculated as

$$\mathbf{A} = \left[\text{DFS}_{2^B} \left\{ \mathcal{T}[z(n)] : x_0 \leq z(n) \leq x_0 + 2^B - 1 \right\} \right]$$

where $\text{DFS}_{2^B}\{\mathcal{T}[z(n)]\}$ generates a column vector containing the 2^B point DFS coefficients of the sequence

$$\left\{ \mathcal{T}[z(n)] : x_0 \leq z(n) \leq x_0 + 2^B - 1 \right\}.$$

By defining the column vector, \mathbf{Y}_l , as the DFS coefficients of $y_l(nT)$ and the column vector, \mathbf{Y}_e , as the DFS coefficients of $y_e(nT)$, the DFS coefficients vectors, \mathbf{Y}_l and \mathbf{Y}_e , can be calculated using $\mathbf{Y}_l = \mathbf{C}\mathbf{A}_l$ and $\mathbf{Y}_e = \mathbf{C}\mathbf{A}_e$, where

$$\mathbf{A}_l = \left[\text{DFS}_{2^B} \left\{ \mathcal{T}_l[z(n)] : x_0 \leq z(n) \leq x_0 + 2^B - 1 \right\} \right]$$

and

$$\mathbf{A}_e = \left[\text{DFS}_{2^B} \left\{ \mathcal{T}_e[z(n)] : x_0 \leq z(n) \leq x_0 + 2^B - 1 \right\} \right].$$

The DFS coefficients vector, \mathbf{Y}_l , describes the spectrum of the DAC's output signal, and the DFS coefficients vector, \mathbf{Y}_e , describes the spectrum of the DAC's harmonic distortion. Because $\mathbf{A} = \mathbf{A}_l + \mathbf{A}_e$, (7) can be written as

$$\mathbf{Y} = \mathbf{C}(\mathbf{A}_l + \mathbf{A}_e) = \mathbf{Y}_l + \mathbf{Y}_e. \quad (8)$$

Because \mathcal{T}_l is a linear transformation, \mathbf{Y}_l can also be written as

$$\mathbf{Y}_l = \bar{q}\mathbf{X}, \quad (9)$$

where \mathbf{X} is a column vector that contains the DFS coefficients of $x(n)$. Substituting (9) into (8), \mathbf{Y} can also be written as

$$\mathbf{Y} = \bar{q}\mathbf{X} + \mathbf{C}\mathbf{A}_e. \quad (10)$$

Three criteria that are used to measure a DAC's performance are SFDR, SDR, and SNDR. A DAC's SFDR can be calculated directly using (6), (7) or (10). To calculate a DAC's SDR, consider the DAC's average signal plus distortion power, P_y , where

$$P_y = \frac{1}{N^2} \mathbf{Y}^H \mathbf{Y} \quad (11)$$

and the superscript H denotes the complex conjugate transpose. Substituting (10) into (11),

$$P_y = \underbrace{\frac{1}{N^2} \bar{q}^2 \mathbf{X}^H \mathbf{X}}_{P_l} + \underbrace{\frac{1}{N^2} \left[2\bar{q} \text{Re}\{\mathbf{X}^H \mathbf{C}\mathbf{A}_e\} + \mathbf{A}_e^H \mathbf{C}^H \mathbf{C}\mathbf{A}_e \right]}_{P_d}$$

where the first term, $\bar{q}^2 \mathbf{X}^H \mathbf{X} / N^2$, is the DAC's average output signal power, P_l , and the second term, $2\bar{q} \text{Re}\{\mathbf{X}^H \mathbf{C}\mathbf{A}_e\} / N^2 + \mathbf{A}_e^H \mathbf{C}^H \mathbf{C}\mathbf{A}_e / N^2$, is the DAC's average distortion power, P_d . Therefore, the DAC's SDR is

$$SDR = \frac{\bar{q}^2 \mathbf{X}^H \mathbf{X}}{2\bar{q} \operatorname{Re}\{\mathbf{X}^H \mathbf{C} \mathbf{A}_e\} + \mathbf{A}_e^H \mathbf{C}^H \mathbf{C} \mathbf{A}_e}. \quad (12)$$

To calculate a DAC's SNDR, consider an input, $x(n)$, which can be written as

$$x(n) = s(n) + w(n)$$

where $s(n)$ is the DAC's input signal and $w(n)$ is the DAC's input noise. Because \mathcal{T}_l is a linear transformation, the DAC's output, $y(nT)$, in (2) can be written as

$$\begin{aligned} y(nT) &= \mathcal{T}[s(n) + w(n)] \\ &= \mathcal{T}_l[s(n)] + \mathcal{T}_l[w(n)] + \mathcal{T}_e[s(n) + w(n)] \end{aligned}$$

and (10) can be written as

$$\mathbf{Y} = \bar{q}\mathbf{S} + \bar{q}\mathbf{W} + \mathbf{Y}_e \quad (13)$$

where \mathbf{S} and \mathbf{W} are vectors that contain the DFS coefficients of $s(n)$ and $w(n)$, respectively. To calculate a DAC's SNDR, consider the DAC's average signal plus noise plus distortion power, P_y , where

$$P_y = \frac{1}{N^2} \mathbf{Y}^H \mathbf{Y}. \quad (14)$$

Substituting (13) into (14),

$$\begin{aligned} P_y &= \frac{1}{N^2} \bar{q}^2 \mathbf{S}^H \mathbf{S} + \frac{1}{N^2} \left\{ \bar{q}^2 \mathbf{W}^H \mathbf{W} + \mathbf{A}_e^H \mathbf{C}^H \mathbf{C} \mathbf{A}_e \right\} \\ &\quad + \frac{1}{N^2} 2\bar{q} \operatorname{Re}\left\{ \bar{q} \mathbf{S}^H \mathbf{W} + \mathbf{S}^H \mathbf{C} \mathbf{A}_e + \mathbf{W}^H \mathbf{C} \mathbf{A}_e \right\}. \end{aligned}$$

where the first term, $\bar{q}^2 \mathbf{S}^H \mathbf{S} / N^2$, is the DAC's average output signal power, and the remaining terms are the DAC's average noise plus distortion power. Therefore, the DAC's SNDR is

$$SNDR = \frac{\bar{q}^2 \mathbf{S}^H \mathbf{S}}{\left(2\bar{q} \operatorname{Re}\left\{ \bar{q} \mathbf{S}^H \mathbf{W} + \mathbf{S}^H \mathbf{C} \mathbf{A}_e + \mathbf{W}^H \mathbf{C} \mathbf{A}_e \right\} + \bar{q}^2 \mathbf{W}^H \mathbf{W} + \mathbf{A}_e^H \mathbf{C}^H \mathbf{C} \mathbf{A}_e \right)}. \quad (15)$$

III. EXAMPLE

Consider a six bit DAC that has linear gradient and uniformly distributed random unit DAC errors where the linear gradient errors vary linearly from +5% to -5% of an LSB and the random errors are uniformly distributed between +2% and -2% of an LSB. This particular example uses the DAC transfer characteristic in Table 1. From the table, it can be seen that $\bar{q} = 2$ and that $\bar{q}x_0 = -63$ where $\bar{q}x_0$ is the DAC's zero input offset. Also for this particular example, the DAC's input sequence is a full scale dithered sinusoid that has a frequency of $7\pi/64$ radians/sample. The dither sequence is a strictly white sequence with a triangular

probability distribution function with support on $(-q, q)$.

Quantizing the dithered sinusoidal input sequence to six bits, (9) can be used to calculate the ideal DAC's power spectral density (PSD) which is shown in Fig. 1. The PSD in Fig.1 is identical to the PSD of the DAC's digital input sequence within a scaling. Using the quantized input and the transfer characteristic in Table 1, (6), (7) or (10) can be used to determine the DAC's output PSD, which is plotted in Fig. 2. Using Fig. 2, the DAC's SFDR is measured to be 38.5 dB. The spectrum, \mathbf{Y}_e , of the DAC's distortion, $y_e(nT)$, can be calculated by substituting \mathbf{A}_e for \mathbf{A} in (7) or substituting \mathcal{T}_e for \mathcal{T} in (6). Fig. 3 shows the PSD of the DAC's distortion. Using (12), the DAC's SDR is 37.5 dB. To calculate the DAC's SNDR, assume that the DAC's input, $x(n)$, has the form, $s(n)+w(n)$, where $s(n)$ is the unquantized sinusoidal input without dither and $w(n)$ is the signal that includes quantization and dither noise. Then using (15), the DAC's SNDR is 33.9 dB. Using Fig. 2, the DAC's signal to noise ratio (SNR) [1] is 36.9 dB. The difference between SNR and SNDR is attributed to the distortion energy at the fundamental frequency as seen in Fig. 3.

IV. SUMMARY

In this paper, a Fourier analysis technique was developed that relates a DAC's transfer characteristic to its output frequency spectrum. In particular, (6), (7) and (10) calculate a DAC's output frequency spectrum given a DAC's transfer characteristic and a particular input signal. The results from (6), (7) and (10) can also be used to determine a DAC's SFDR. Using the DAC's INL instead of its transfer characteristic, (6), (7) and (10) can also be used to calculate a DAC's harmonic distortion. Also for a given transfer characteristic and a particular input signal, a DAC's SDR and SNDR can be calculated using (12) and (15).

REFERENCE

- [1] D.H. Sheingold, ed., *Analog-Digital Conversion Handbook*. Englewood Cliffs, NJ: Prentice-Hall, 1986.

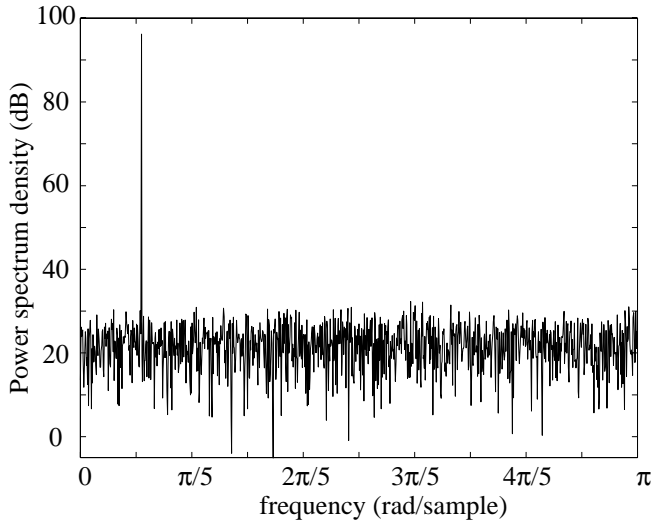


Figure 1. Power spectral density plot of the output, $y_l(nT)$, which is the linear part of the Example's nonlinear DAC. The DAC's input is a full scale dithered sine wave that has a frequency of $7\pi/64$ rad/sample.

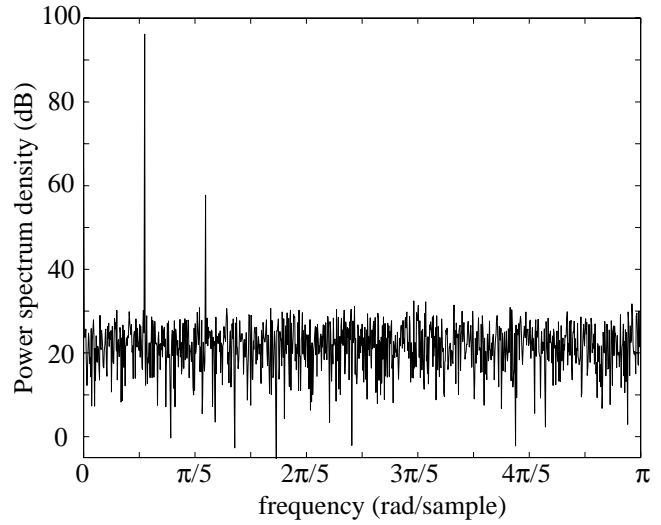


Figure 2. Power spectral density plot of the output, $y(nT)$, of the Example's nonlinear DAC. The DAC's input is a full scale dithered sine wave that has a frequency of $7\pi/64$ rad/sample.

Table 1. Transfer characteristic, \mathcal{T} , of the Example's nonlinear DAC

-63.0000	-32.1730	-0.5570	31.9130
-61.1280	-30.2250	1.4510	33.9500
-59.2000	-28.2650	3.4660	36.0120
-57.2850	-26.3160	5.4770	38.0800
-55.3820	-24.3640	7.5210	40.1160
-53.4290	-22.4020	9.5330	42.1640
-51.5240	-20.4360	11.5490	44.2130
-49.5930	-18.4910	13.5490	46.2620
-47.6700	-16.5070	15.6080	48.3660
-45.7320	-14.5100	17.6260	50.4450
-43.8240	-12.5340	19.6770	52.5200
-41.8700	-10.5640	21.6870	54.6080
-39.9360	-8.5550	23.7450	56.7140
-37.9880	-6.5550	25.7680	58.7890
-36.0630	-4.5570	27.8110	60.9120
-34.1180	-2.5520	29.8690	63.0000

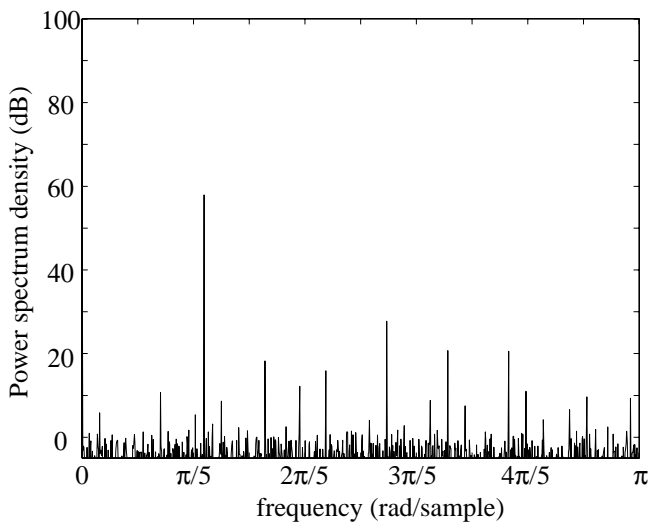


Figure 3. Power spectral density plot of the harmonic distortion, $y_e(nT)$, of the Example's nonlinear DAC. The DAC's input is a full scale dithered sine wave that has a frequency of $7\pi/64$ rad/sample.