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A Reachability based Stability Analysis for Switching Systems

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A Reachability based Stability Analysis for Switching Systems*

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Abstract

In this paper we discuss the practical stability issue of a switching system in terms of solving a containability problem and an attraction problem. A novel computational procedure based on nonlinear programming is presented to compute a containable region, in which each trajectory from inside cannot move out under a given single-step-lookahead control policy. Then we discuss how to decide whether the obtained containable region is finitely reachable from a point outside the region.

1 Introduction

This paper considers a practical stability problem for a class of multi-mode nonlinear systems. Let I be a finite index set. Suppose the state space R^n is partitioned into a finite collection of disjoint nonempty sets $\{X_i \subseteq R^n | i \in I\}$, where each set X_i is called a *local mode* and defined as the solution set of a collection of known inequalities $\phi_{ij}(x) \leq 0$ or $\phi_{ij}(x) < 0$ with $1 \leq j \leq m_i$, where each $\phi_{ij}(x)$ is differentiable in x (for the latter nonlinear optimization purpose). For example, if each ϕ_{ij} is a linear map then X_i is a polyhedron. Each X_i is associated with a discrete-time dynamic model

$$x(k+1) = f_i(x(k), u(k)) \tag{1}$$

where $x(k) \in X_i$, $x(k+1) \in R^n$, $u(k) \in U_i \subset R^m$ and f_i is differentiable over x for each fixed $u \in U_i$. Each control action set U_i $(i \in I)$ is finite and not necessarily share elements with some other control action set U_j $(i \neq j)$. A multi-mode system belongs to the class of *switched systems* [4, 1, 2], where each control input of $\cup_{i\in I} U_i$ corresponds to a potential transition between two modes. Equation (1) can be described as an impulse-driven dynamic behavior. At each state $x(k) \in R^n$, if $x(k) \in X_i$ for some $i \in I$ then we can apply "impulsive force" (i.e. $u(k)$) on it, where the impulse force only exists at the time instant k. The resulting state $x(k+1)$ may still be in X_i or may not. Such an "impulse-driven" interpretation makes it easy for us to describe the dynamic behavior, that crosses boundary of different modes X_i and X_j $(i \neq j)$, as simply an autonomous mode change.

In a switching system we consider a safety control problem which is stated as follows. Given a safe state $x_s \in R^n$ and a set of initial states $X_o \subseteq R^n$, design a controller S that can drive the system from any state in X_o to a small neighborhood of x_s in a finite number of time steps using switching actions from a finite set $\cup_{i\in I} U_i$, and keep the system's trajectory within that small neighborhood. The control strategy proposed in this paper is conceptually similar to the model predictive control [9, 11, 12, 7] in which a limited time forecast of the process behavior at each state is optimized according to a given criterion over the set of inputs. Also related to our work is the online limited lookahead supervision of discrete event systems (DES) [5]. More precisely, the selection of the control action at a specific state is based on a distance map that defines how close the next state is to the safe state x_s. A control action

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that can take the system to a state closest to the safe state x_s will be chosen. In order to guarantee that such an online control strategy works, two questions need to be answered: (1) Does there exist a neighborhood of x_s such that any trajectory inside it cannot move out under the given control policy? (2) Does there exist a set of initial states from which a trajectory can move into the neighborhood mentioned in (1) within a finite number of steps?

The first question is closely related to the containability problem described in the literature, e.g. [15] and [8]. It is a practical stability problem, which states that given a perturbation at the origin, the trajectory will be confined in a small neighborhood of the origin. It is different from the classic Lyapunov stability in that the small neighborhood may not become arbitrarily small even though the perturbation can be arbitrarily small, considering that there are only finite number of control actions to stabilize the system. To solve the practical stability problem, a Lyapunov-like approach [8] is adopted in the literature. Nevertheless, it is well known that there is no general way to construct an appropriate Lyapunov-like function. In this paper, we propose a new method in which the containability problem is converted into a nonlinear programming problem. The main advantage of the proposed approach is its computation efficiency, namely given any nonlinear system, where the dynamics $f_i(x, u)$ $(i \in I)$ is differentiable in x (so that nonlinear programming can be applied), we can simply supply $f(x, u)$ to the proposed computational procedure to compute a special containable region. On the other hand, nonlinear programming cannot always guarantee global optimality. But this can be compensated by choosing as many seeds during computation as possible.

The second question is closely related to the attraction problem. A standard attraction problem can be described as follows: given a locally exponentially stable fixed point $x^* \in R^n$, find the set of points which are attracted to x^*. In this paper, instead of a single desired point x^*, we want to find a set of points that are attracted to a given containable region. Although not much publications on the standard attraction problem for discrete-time systems, there are plenty of them on continuous-time systems, e.g. [16, 10, 6, 13]. In these approaches, [10, 13] require the numerical solution of several Hamilton-Jacobi-Bellman equations, which can be very computationally demanding. [6] needs just one such solution, but requires some knowledge about the local behavior around x^* in order to avoid discontinuities in the optimal value functions causing numerical problems. [16] utilizes the traditional Lyapunov approach. Nevertheless, the finite reachability issue is not explicitly addressed, instead they deal with asymptotic stability. Of course, if indeed such asymptotic stability is achievable then a small neighborhood of x^* is finitely reachable. But in this paper, due to a finite number of control actions, asymptotic stability does not exist in the discrete-time system described by equation (1). Although it may be possible to modify the above approach to handle the practical attraction problem considered in this paper, we propose a new and simple way to judge whether the containable region is finitely reachable.

This paper is organized as follows. In Section II we first introduce a practical stability problem in a single-mode system, and propose a computational procedure to solve it. Then in Section III the theory is extended to handle a multi-mode system. After discussion on computing the containable region in parameter-uncertain systems in Section IV, we draw conclusions in Section V.

2 A practical stability problem in a Single-mode system

Suppose the index set I is a singleton. Then equation (1) describes the dynamics of a single-mode system

$$x(k+1) = f(x(k), u(k))$$

where $x(k) \in R^n$, $u(k) \in U = \{u_1, \cdots, u_p\} \subset R^m$ for some $p \in \mathbb{N}^+$ and $f : R^n \times U \to R^n$ is differentiable in R^n for each fixed $u \in U$. Suppose the origin is the desirable state. For each initial state $x(0) \in R^n$ it is desirable to move the trajectory to a small closed neighborhood $D \subset R^n$ of the origin, which is called a *desired region*, after a finite number of transient transitions, then keep the trajectory inside D forever. To this end, in a *one-step-lookahead* approach the online control policy is defined as follows: for each $x \in R^n$ pick $u^* \in U$ such that

$$\|f(x, u^*)\| = \min_{u \in U} \|f(x, u)\| \tag{2}$$

where $\|\cdot\|$ means l_2-norm. To discuss the effectiveness of the above control policy the following questions are addressed.

1. Containability problem: Determine the existence of a bounded closed subset $A \subseteq D$, where

$$(\forall x \in A)\, f(x, u^*) \in A \tag{3}$$

   A closed set $A \subseteq R^n$ satisfies (3) is called a *containable region* under control policy u^*.

2. Practical attraction problem: Decide if a containable region in D is finitely reachable from any initial state $x(0) \in R^n$ under the given control policy u^*.

In [15] the authors proposed a similar concept of containability. But their problems is about feedback control with finite communication bandwidth, and a linear continuous time system is under consideration. The practical attraction problem has also been mentioned in the literature about practical stability [8]. Both the containability problem and the attraction problem may be solved by using the indirect Lyapunov method, namely to construct a discrete-time Lyapunov function and determine the stability region for the containability problem and the asymptotical stability region for the attraction problem. Nevertheless, it is well known that there is no general way to construct a Lyapunov function. In this paper an alternatively method is proposed to solve the containability problem and the practical attraction problem by using nonlinear programming.

2.1 Containable Region

For each control action $u_i \in U$, let

$$W_i := \left\{ x \in R^n \;\middle|\; \|f(x, u_i)\| < \|x\| \right\}$$

That is, W_i is the set of states at which the control action u_i can bring the system closer to the origin. Let

$$Q := \bigcup_{i=1}^{m} W_i$$

Intuitively, Q is the set of states where there exists a control action that can bring the system closer to the origin. Let R^+ be the set of all nonnegative real numbers. For an $r \in R^+$ write $B(r)$ for the closed ball in R^n centered at the origin with the radius r, and $\partial B(r)$ for the boundary of $B(r)$, namely

$$\partial B(r) := \left\{ x \in B(r) \middle| \; \|x\| = r \right\}$$

Let $\overline{Q}$ be the complement of Q with respect to R^n, i.e. $\overline{Q} := R^n - Q$. Let

$$r^* = \max_{x \in \overline{Q}} \|f(x, u^*)\| = \max_{x \in \overline{Q}} \min_{u \in U} \|f(x, u)\| \tag{4}$$

By the definition of $\overline{Q}$,

$$(\forall x \in \overline{Q})(\forall u \in U)\; \|x\| \leq \|f(x, u)\|$$

Particularly,

$$\|x\| \leq \min_{u \in U} \|f(x, u)\| \leq \max_{x' \in \overline{Q}} \min_{u \in U} \|f(x', u)\| = r^*$$

Thus, $\overline{Q} \subseteq B(r^*)$. Furthermore, we have the follows.

Proposition 1 $B(r^*)$ is a containable region under the given control policy. Furthermore, if there exists another containable region $B(r)$ with $\overline{Q} \subseteq B(r)$, then $r^* \leq r$.

Proof: For each $x \in B(r^*)$, either $x \in Q$ or $x \in \overline{Q}$. If $x \in Q$ then by definition of Q,

$$\|f(x,u^*)\| = \min_{u\in U} \|f(x,u)\| \leq \|x\| \leq \hat{r}^*$$

If $x \in \overline{Q}$ then we have

$$\|f(x,u^*)\| = \min_{u\in U} \|f(x,u)\| \leq \max_{x'\in\overline{Q}} \min_{u\in U} \|f(x',u)\| \leq r^*$$

In both cases, $f(x,u^*) \in B(r^*)$. So $B(r^*)$ is a containable region. Suppose there is another closed ball $B(r)$ which is a containable region with $\overline{Q}$ inside. Then by the definition of containable region,

$$(\forall x \in \overline{Q})\, \|f(x,u^*)\| = \min_{u\in U} \|f(x,u)\| \leq r$$

Thus,

$$r^* = \max_{x\in\overline{Q}} \min_{u\in U} \|f(x,u)\| \leq r$$

as required. ■

Computing r^* is a max-min problem, which can be converted to the following nonlinear programming problem.

$$\begin{array}{ll} \text{maximize} & r^* \\ \text{subject to} & (\forall u \in U)\, ||f(x,u)||^2 \geq (r^*)^2 \\ & r^* \geq 0 \\ & (\forall u \in U)\, ||f(x,u)||^2 \geq ||x||^2 \end{array}$$

Since for each fixed $u \in U$, $f(x,u)$ is differentiable in x, so is $\|f(x,u)\|^2$ (but $\|f(x,u)\|$ may not be). Thus the above nonlinear programming problem is well defined [14]. Let $D \subset R^n$ be a bounded closed neighborhood of the origin. If $B(r^*) \subseteq D$ then there is a containable region inside D, namely the system's trajectory can be confined within a small neighborhood of the desired state x_s. But it is only a sufficient condition, which has a similar role as a Lyapunov-like function in standard stability theory. Nevertheless, finding $B(r^*)$ is much easier than finding an appropriate Lyapunov function, as indicated above. However, nonlinear programming is not guaranteed to converge to the globally optimal solution unless it is a convex optimization. Nevertheless, in practical applications we can select as many initial seeds as possible during iteration in order to increase our confidence in the correctness of r^*. Furthermore, it is well known that the dual problem of the above primal optimization problem is a convex problem [3] and the dual optimal solution is an upper bound of the optimal solution of the primal problem. So after the practical computation of the primal problem converges, which may not necessarily be the global optimal solution, we can compute the dual optimal solution and determine the duality gap. If the gap is small then we know the solution we have obtained for the primal problem is a good estimate of the unknown global optimal solution. If the gap is large, then for the conservativeness reason we can take the dual optimal solution as the radius of the containable region. On the other hand, if it is known that $\|f(x,u)\|$ is a convex function in x and $\overline{Q}$ is a convex region, then the globally optimal solution is guaranteed to be found. For example, if $f(x,u)$ is linear, say $f(x,u) = Ax + Bu$, then clearly $\|f(x,u)\|$ is a convex function. If all eigenvalues of A^tA is in a unit circle, then from standard convex analysis we know that, for each $u \in U$ the region $\left\{x \in R^n \middle| \|x\| \leq \|f(x,u)\|\right\}$ is an ellipsoid, which implies that

$$\overline{Q} = \bigcap_{u\in U} \left\{x \in R^n \middle| \|x\| \leq \|f(x,u)\|\right\}$$

is convex. Next, we discuss the finite reachability issue.

2.2 Finite Reachability of Containable Regions

Let $\mathbb{N}$ be the set of natural numbers. A closed ball $B(r) \subseteq R^n$ is *finitely reachable under the given control policy* if

$$(\forall x_0 \in R^n)(\exists N(x_0, r) \in \mathbb{N})\, x(N(x_0, r)) \in B(r) \tag{5}$$

where $x(N(x_0, r))$ is the state at time instant $N(x_0, r)$ under the given control policy with the initial state x_0. In practice, the initial state of a system may not be in the desired region D. So we want to know whether or not a containable region in D can be finitely reached from an initial state outside D. To this end, we have the following.

Lemma 1 Let $g : R^n \to R$ be a function such that

$$(\forall x \in R^n)\, g(x) := \|x\| - \|f(x, u^*)\| = \|x\| - \min_{u \in U} \|f(x, u)\|$$

Then g is a continuous function.

Proof: By definition of g we have

$$g(x) = \max_{u \in U}(\|x\| - \|f(x, u)\|)$$

Since $f(x, u)$ is differentiable in x when u is fixed, we get that $h(x, u) := \|x\| - \|f(x, u)\|$ is continuous in x given u. Thus, for each $\epsilon > 0$ there exists a $\delta_u > 0$ such that

$$\|x - x'\| < \delta_u \Rightarrow |h(x, u) - h(x', u)| < \epsilon$$

Let

$$\delta := \min_{u \in U} \delta_u$$

Since U is a finite set, we have $\delta > 0$. For any $x' \in R^n$ with $\|x - x'\| < \delta$,

$$|g(x) - g(x')| = |\max_{u \in U} h(x, u) - \max_{u' \in U} h(x', u')| \leq \max_{u \in U} |h(x, u) - h(x', u)| < \epsilon$$

So g is continuous, as required. ■

Proposition 2 Let $B(r^*)$ be the containable region with $r^* \in R^+$ obtained from (4). If $\partial B(r^*) \subseteq Q$ then $B(r^*)$ is finitely reachable under the given control policy.

Proof: We need to show that

$$(\forall x_0 \in R^n)(\exists N(x_0, r^*) \in \mathbb{N})\, x(N(x_0, r^*)) \in B(r^*)$$

Suppose otherwise. Then there exists an initial state x_0 such that no such $N(x_0, r^*)$ exists. Clearly $x_0 \notin B(r^*)$, and furthermore

$$(\forall k \in \mathbb{N})\, \|x(k)\| > r^*$$

By definition of r^* and the assumption $\partial B(r^*) \subseteq Q$, we have

$$(\forall k \in \mathbb{N})\, x(k) \in Q$$

By definition of Q, for each $k \in \mathbb{N}$ we have

$$g(x(k)) = \|x(k)\| - \|f(x(k), u(k)^*)\| = \|x(k)\| - \|x(k+1)\| > 0$$

Since $\sum_k g(x(k)) \leq \|x_0\| - r^*$, by Bolzano-Weierstrass Theorem, there exists a convergence sequence

$$\{g(x(j_k))|k \in \mathbb{N}\} \subseteq g(B(\|x_0\|) - B(r^*))$$

such that

$$\lim_{k\to+\infty} g(x(j_k)) = 0$$

On the other hand, by Lemma 2.1 g is continuous in x. Clearly the set

$$\Omega := B(\|x_0\|) - (B(r^*) - \partial B(r^*))$$

is compact. As we know a continuous function maps a compact set to a compact set. So $g(\Omega)$ is also compact. Thus,

$$\lim_{k\to+\infty} g(x(j_k)) = 0 \in g(\Omega)$$

namely there exists $x \in \Omega$ such that $g(x) = 0$. But $\Omega \in Q$ and by definition of Q we have

$$(\forall x' \in \Omega)\, g(x') > 0$$

which contradicts $g(x) = 0$. ■

Proposition 2.2 says that, given a desired region D, if the containable region $B(r^*) \subset D$ has its boundary set $\partial B(r^*) \subseteq Q$, then $B(r^*)$ is finitely reachable from any initial state $x \in R^n$. Let

$$\hat{r} := \max_{x\in\overline{Q}} \|x\| \tag{6}$$

which is solved by the following nonlinear programming.

$$\begin{array}{ll} \text{maximize} & \|x\|^2 \\ \text{subject to:} & (\forall u \in U)\, \|x\|^2 \leq \|f(x,u)\|^2 \end{array}$$

Clearly $\overline{Q} \subseteq B(\hat{r})$. Then we have the following result.

Corollary 1 Let r^* be obtained from (4) and $\hat{r}$ from (6). Then $r^* \geq \hat{r}$. If $r^* > \hat{r}$ then $B(r^*)$ is a finitely reachable containable region.

Proof: By the definition of $\overline{Q}$ we have

$$r^* = \max_{x\in\overline{Q}} \|f(x,u^*)\| \geq \max_{x\in\overline{Q}} \|x\| = \hat{r}$$

By Prop. 2.1, $B(r^*)$ is a containable region. If $r^* > \hat{r}$ then clearly $\partial B(r^*) \subseteq Q$ which, by Prop. 2.2, implies that $B(r^*)$ is finitely reachable. ■

2.3 Example

As an illustration, consider the following real-life example - a two-tank system, which is shown in Figure 1. There are two valves in the system, each of which can be either open or closed. The total height of each tank is 1.0 meter. Its simplified one-mode dynamics is described by the following discrete-time equation:

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) - b v_{12}(k)\sqrt{x_1(k) - 0.3} + x_1(k) \\ b v_{12}(k)\sqrt{x_1(k) - 0.3} - a_2 + x_2(k) \end{bmatrix}$$

where $u(k) = [v_1(k)\ v_{12}(k)]$ with $v_1(k), v_{12}(k) \in \{0,1\}$, $0.3 \leq x_1(k) \leq 1.0$, $0 \leq x_2(k) \leq 0,3$. The desired state is set as $x_s = [0.60\ 0.10]$. We compute r^* and $\hat{r}$ by using nonlinear programming with the control set

$$U = \{u_1 = [0,0], u_2 = [0,1], u_3 = [1,0], u_4 = [1,1]\}$$

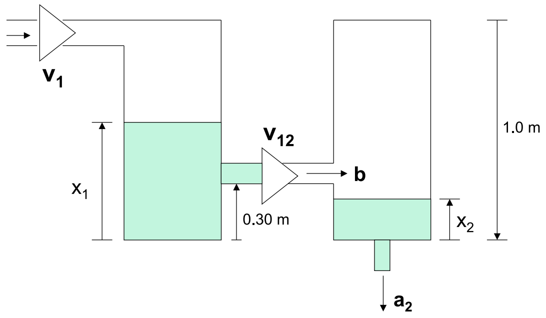

Figure 1: Two-Tank System Physical Model

To speed up computation, practical constraints on x_1 and x_2 are added to the nonlinear programming. For example, to compute $\hat{r}$ let

$$\begin{aligned} \hat{r}^2 &:= \max_x \|x - x_s\|^2 \\ \text{subject to: } \|f(x,u_1) - x_s\|^2 &\geq \|x - x_s\|^2 \\ \|f(x,u_2) - x_s\|^2 &\geq \|x - x_s\|^2 \\ \|f(x,u_3) - x_s\|^2 &\geq \|x - x_s\|^2 \\ \|f(x,u_4) - x_s\|^2 &\geq \|x - x_s\|^2 \\ x_1 &\geq 0.30 \\ 1.0 &\geq x_1 \\ 1.0 &\geq x_2 \\ x_2 &\geq 0 \end{aligned}$$

A desired region is

$$D := \{(x_1, x_2) \in R^2 | 0.59 \leq x_1 \leq 0.61 \,\&\, 0.09 \leq x_2 \leq 0.11\}$$

We perform two experiments.

Case 1: Set $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$. Since this is a two-dimensional problem, it can be solved geometrically. The symbolic solver in MATLAB is used to depicts the region of $\overline{Q}$ by computing the boundaries,

Boundary 1:	$\|f(x,u_1) - x_s\| = \|x - x_s\|$
Boundary 2:	$\|f(x,u_2) - x_s\| = \|x - x_s\|$
Boundary 3:	$\|f(x,u_3) - x_s\| = \|x - x_s\|$
Boundary 4:	$\|f(x,u_4) - x_s\| = \|x - x_s\|$
Boundary 5:	$0.30 \leq x_1 \leq 1.0$
Boundary 6:	$0 \leq x_2 \leq 1.0$

The left picture in Figure 2 depicts those boundaries for each control action $u \in U$. The label on each line represents which control action is taken, e.g. the line with the label [0,1] meas that that line is the boundary with control action $[v_1 = 0, v_{12} = 1]$. Each boundary is a hyper surface which divides the space R^n into two part. From Figure 2 one can see that in the working space $[0.3, 1.0] \times [0, 1.0]$, $\overline{Q}$ is the smallest triangle that contains x_s, which is surrounded by the attraction region Q. The right picture in Figure 2 is an amplified picture for the region $\overline{Q}$, which clearly shows that $x_s \in \overline{Q}$. Since $\overline{Q}$ is bounded, it is possible to compute r^* and $\hat{r}$ by using the nonlinear programming toolbox in MATLAB. The initial pick is $x = (0.55, 0.09)$ for the iteration in nonlinear programming. It turns out that $r^* = 0.0253$. The

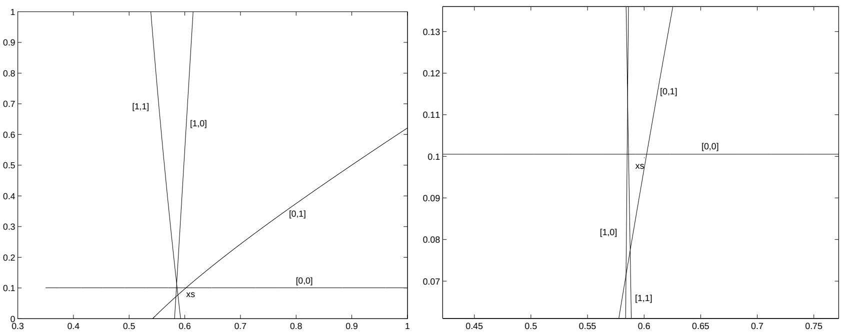

Figure 2: $\overline{Q}$ with $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$

dual optimal solution indicates that $r^* = 0.0253$ is indeed the global optimal solution of the primal problem because no duality gap exists. It is easy to check that the containable region $B(r^*, x_s)$, which is centered at x_s with radius r^*, is not entirely contained in the desired region D. So it is not clear whether there is a containable region in D.

Case 2: $a_1 = 0.003$, $a_2 = 0.0001$ and $b = 0.0005$. By using the similar procedure we get that $\overline{Q}$ is bounded. Figure 3 depicts what the region $\overline{Q}$ looks like, where the left picture is for the whole working space and the right one is the amplified picture for $\overline{Q}$. Compared with Figure 2, we can see that the

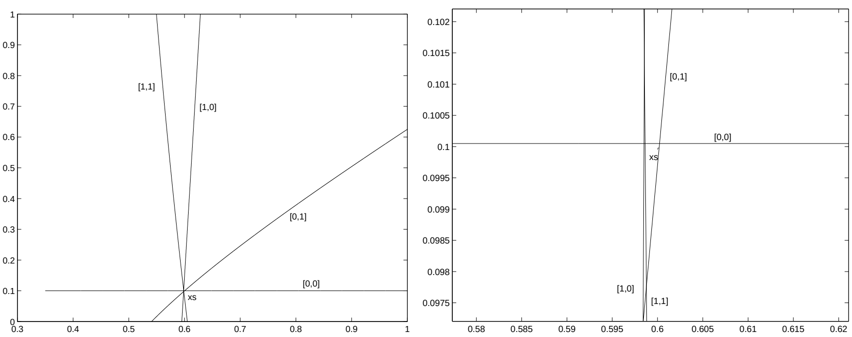

Figure 3: $\overline{Q}$ with $a_1 = 0.003$, $a_2 = 0.0001$ and $b = 0.0005$

region $\overline{Q}$ for the second setup is much smaller than the first one. So we expect that a much smaller containable region can be achieved in the second setup. Indeed, the following results confirm it. To compute r^* and $\hat{r}$, the initial pick is $x = (0.55, 0.09)$ for the iteration in nonlinear programming. It turns out that $r^* = 0.0025$. Again the dual optimal solution confirms that $r^* = 0.0025$ is the global optimal solution of the primal problem because no duality gap exists. In this case, it is easy to check that the containable region $B(r^*, x_s)$ is entirely inside the desired region D. So clearly there is a containable region in D. We further get that $\hat{r} = 0.0025$. Since $r^* = \hat{r}$ and $B(r^*) \subset D$, there is another containable region $B(r^* + \epsilon, x_s) \subset D$, with a sufficiently small $\epsilon > 0$. Since $\partial B(r^* + \epsilon, x_s) \in Q$, by Corollary 2.1 $B(r^* + \epsilon, x_s)$ is finitely reachable. Figure 4 depicts the simulation of the two-tank system under the given control policy with the initial state $x_0 = (0.6, 0.1)$, where the left picture depicts x_1, x_2 versus the time t, and the right picture depicts x_2 versus x_1. from which we can see that the trajectory is bounded in a small neighborhood of x_s within the desired region D after a finite number of transitions.

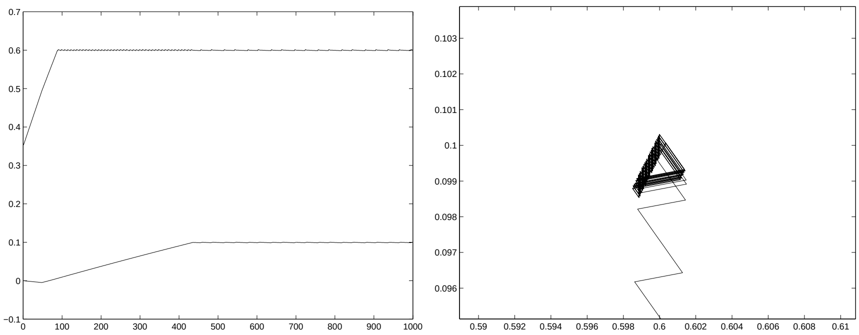

Figure 4: simulation with $a_1 = 0.003$, $a_2 = 0.0001$ and $b = 0.0005$

3 Containable region for a multi-mode system

In this section we will extend our theory to a multi-mode system. Suppose the system dynamics is described in equation (1). Suppose the origin is the desired state. Then the one-step lookahead control policy is defined as follows. Let $\phi : R^n \to I$ be an index map such that

$$(\forall x \in R^n)\, \phi(x) := i \text{ if } x \in X_i$$

Since $\{X_i | i \in I\}$ is a partition of R^n, ϕ is well defined. For each state $x \in R^n$, pick $u^* \in U_{\phi(x)}$ such that

$$||f_{\phi(x)}(x, u^*)|| = \min_{u \in U_{\phi(x)}} ||f_{\phi(x)}(x, u)|| \tag{7}$$

As before, our objective is to decide whether in a given set $D \subset R^n$ there is a containable region $S \subseteq D$ such that

$$(\forall x \in S)\, f_{\phi(x)}(x, u^*) \in S$$

To this end, we follow the procedure used in previous sections. For each $i \in I$ and $u \in U_i$ let

$$W_u := \left\{ x \in X_i \middle| ||f_i(x, u)|| < ||x|| \right\}$$

and

$$Q_i := \bigcup_{u \in U_i} W_u$$

So Q_i is the set of all states in X_i where there is a control action $u \in U_i$ that can bring the system's trajectory closer to the desired state - the origin. Let

$$Q := \bigcup_{i \in I} Q_i$$

As before, $\overline{Q} := R^n - Q$. Let $\hat{X}_i := X_i \cup \partial X_i$ be the closure of X_i, where ∂X_i is the boundary.

$$r^* := \max_{i \in I} \max_{x \in \overline{Q} \cap \hat{X}_i} ||f_i(x, u^*)|| = \max_{i \in I} \max_{x \in \overline{Q} \cap \hat{X}_i} \min_{u \in U_i} ||f_i(x, u)|| \tag{8}$$

Proposition 3 The closed ball $B(r^*)$ with r^* computed in (8) is a containable region.

Proof: For each $x \in B(r^*)$, suppose $i = \phi(x)$. If $x \in Q_i$ then by definition of Q_i we have

$$||f_i(x, u^*)|| < ||x|| \le r^*$$

If $x \notin Q_i$ then $x \in \overline{Q} \cap X_i$. By (8) we have

$$||f_i(x, u^*)|| \le \max_{x' \in \overline{Q} \cap X_i} ||f_i(x', u^*(x'))|| \le \max_{x' \in \overline{Q} \cap \hat{X}_i} ||f_i(x', u^*(x'))|| \le r^*$$

In both cases we have $f_i(x, u^*) \in B(r^*)$, as required. ■

To compute r^* we first determine the set

$$J := \{i \in I | \overline{Q} \cap \hat{X}_i \neq \varnothing\}$$

so that later when we propose a nonlinear programming approach to compute r^*, each local optimization problem has admissible solutions. Based on the definition of $\overline{Q}$ we can see that

$$\overline{Q} \cap \hat{X}_i = \{x \in \hat{X}_i | (\forall u \in U_i)\, ||f_i(x, u)|| \ge ||x||\}$$

We know that

$$\hat{X}_i := \{x \in R^n | \phi_{ij}(x) \le 0 \text{ with } 1 \le j \le m_i\}$$

So $\overline{Q} \cap \hat{X}_i \neq \varnothing$ if and only if the following minimization problem attains zero as the solution.

$$\begin{array}{ll} \text{minimize} & ||x - y||^2 \\ \text{subject to} & (\forall j : 1 \le j \le m_i)\, \phi_{ij}(x) \le 0 \\ & (\forall u \in U_i)\, ||y||^2 \le ||f_i(y, u)||^2 \end{array}$$

It is well known in the nonlinear optimization theory [3] that computing r^* in (8) can be converted into the following nonlinear programming problem.

$$\begin{array}{ll} \text{maximize} & z_i \text{ with } i \in J \\ \text{subject to} & (\forall u \in U_i)\, ||f_i(x, u)||^2 \ge z_i^2 \\ & z_i \ge 0 \\ & (\forall u \in U_i)\, ||f_i(x, u)||^2 \ge ||x||^2 \\ & (\forall j : 1 \le j \le m_i)\, 0 \ge \phi_{ij}(x) \\ \text{final objective} & r^* := \max_{i \in J} z_i \end{array}$$

If the above optimization problem is not convex, then we need to solve its dual problem to decide the duality gap and obtain a conservative solution of r^*, e.g. when there is a duality gap exists, we can pick the dual optimal solution as the value of r^* which is larger than the primal optimal solution.

To achieve a finite reachable containable region, we follow a similar procedure as described in Section II.B. For each $i \in I$ let

$$g_i : \hat{X}_i \to R : x \mapsto g_i(x) := ||x|| - \min_{u \in U_i} ||f_i(x, u)||$$

By Lemma 2.1 we know that g_i is continuous on $\hat{X}_i = X_i \cup \partial X_i$. Notice that X_i itself may not be necessarily closed.

Definition 1 Let $r \in \mathbb{N}^+$. The system, whose dynamics is described in (1), is *r-boundary active* if $(\forall i \in I)(\exists \epsilon_i \in R^+)(\forall x \in \partial X_i)\, ||x|| \ge r \Rightarrow g_i(x) \ge \epsilon_i$. □

The r-boundary activity can be verified through the following minmax problem

$$(\forall i \in I)\, J_i := \min_{x \in \partial X_i,\, ||x|| \ge r} \max_{u \in U_i} ||x|| - ||f_i(x, u)||$$

If $J_i > 0$ for each $i \in I$, then the system is r-boundary active. The minmax problem can be converted into the following nonlinear programming:

$$\begin{array}{rl} \text{minimize} & \epsilon_{ij} \text{ with } i \in I \text{ and } 1 \leq j \leq m_i \\ \text{subject to} & (\forall u \in U_i)\, ||x|| - ||f_i(x,u)|| \leq \epsilon_{ij} \\ & ||x|| \geq r \\ & \phi_{ij} = 0 \\ & (\forall p : 1 \leq p \leq m_i, p \neq j)\, \phi_{ip}(x) \leq 0 \\ \text{objective} & \epsilon_i = \max\limits_{j:1 \leq j \leq m_i} \epsilon_{ij} \end{array}$$

If $\epsilon_i > 0$ for each $i \in I$, then the system is r-boundary active.

Proposition 4 Suppose r^* is computed from (8), and the system is r^*-boundary active. If $\partial B(r^*) \in Q$ then $B(r^*)$ is finitely reachable under the given control policy.

Proof: We follow a similar procedure as in Proposition 2.1 to show that

$$(\forall x_0 \in R^n)(\exists N(x_0, r^*) \in \mathbb{N})\, x(N(x_0, r^*)) \in B(r^*)$$

Suppose otherwise. Then there exists an initial state x_0 such that no such $N(x_0, r^*)$ exists. By Proposition 3.1, clearly $x_0 \notin B(r^*)$. Furthermore

$$(\forall k \in \mathbb{N})\, \|x(k)\| > r^*$$

By definition of r^* and the assumption $\partial B(r^*) \subseteq Q$, we have

$$(\forall k \in \mathbb{N})\, x(k) \in Q$$

Since $Q = \cup_{j \in I} Q_j$ and I is finite, there is an $i \in I$ such that X_i contains an infinite number of states along the trajectory, say $\Delta_i = \{x(k_l) | l \in \mathbb{N}\} \subseteq \{x(k) | k \in \mathbb{N}\}$. By definition of Q_i, for each $l \in \mathbb{N}$ we have

$$g_i(x(k_l)) = \|x(k_l)\| - \|f_i(x(k_l), u(k_l)^*)\| = \|x(k_l)\| - \|x(k_l + 1)\| > 0$$

Since

$$\sum_{x \in \Delta_i} g_i(x) \leq \sum_{k=0}^{+\infty} ||x(k)|| - ||x(k+1)|| \leq \|x_0\| - r^*$$

by Bolzano-Weierstrass Theorem, there exists a convergence sequence

$$\{g_i(x(j_l)) | x(j_l) \in \Delta_i \,\&\, l \in \mathbb{N}\} \subseteq g_i((B(\|x_0\|) - B(r^*)) \cap (X_i \cup \partial X_i))$$

such that

$$\lim_{l \to +\infty} g_i(x(j_l)) = 0$$

Since g_i is continuous on $X_i \cup \partial X_i$, and the set

$$\Omega_i := (B(\|x_0\|) - (B(r^*) - \partial B(r^*))) \cap (X_i \cup \partial X_i)$$

is compact, we have that

$$\lim_{l \to +\infty} g_i(x(j_l)) = 0 \in g_i(\Omega_i)$$

namely there exists $x \in \Omega_i$ such that $g_i(x) = 0$. But by definition of Q_i and the r^*-boundary activity of the system, we have

$$(\forall x' \in \Omega_i)\, g_i(x') > 0$$

which contradicts $g_i(x) = 0$. ■

As an illustration, let us revisit the two-tank system depicted in Figure 1. This time we set the desired state as $x_s = [0.60, 0.30]$. The desired region is

$$D := \{(x_1, x_2) \in R^2 | 0.59 \leq x_1 \leq 0.61 \,\&\, 0.29 \leq x_2 \leq 0.31\}$$

We have the following multi-mode model:

(1) Mode 1: $0 \leq x_1 < 0.3$ and $0 \leq x_2 < 0.3$

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) + x_1(k) \\ 0 \end{bmatrix}$$

where $u(k) = [v_1(k) \; v_{12}(k)]$ with $v_1(k), v_{12}(k) \in \{0, 1\}$.

(2) Mode 2: $0 \leq x_1 < 0.3$ and $0.3 \leq x_2 \leq 1.0$

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) + b v_{12}(k) \sqrt{x_2(k) - 0.3} + x_1(k) \\ -b v_{12}(k) \sqrt{x_2(k) - 0.3} - a_2 + x_2(k) \end{bmatrix}$$

(3) Mode 3: $0.3 \leq x_1 \leq 1.0$ and $0 \leq x_2 < 0.3$

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) - b v_{12}(k) \sqrt{x_1(k) - 0.3} + x_1(k) \\ b v_{12}(k) \sqrt{x_1(k) - 0.3} - a_2 + x_2(k) \end{bmatrix}$$

(4) Mode 4: $0.3 \leq x_1 < x_2 \leq 1.0$

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) + b v_{12}(k) \sqrt{x_2(k) - x_1(k)} + x_1(k) \\ -b v_{12}(k) \sqrt{x_2(k) - x_1(k)} - a_2 + x_2(k) \end{bmatrix}$$

(5) Mode 5: $0.3 \leq x_2 \leq x_1 \leq 1.0$

$$x(k+1) = f(x(k), u(k)) = \begin{bmatrix} a_1 v_1(k) - b v_{12}(k) \sqrt{x_1(k) - x_2(k)} + x_1(k) \\ b v_{12}(k) \sqrt{x_1(k) - x_2(k)} - a_2 + x_2(k) \end{bmatrix}$$

Similar as we did in Section III, here we use two sets of parameters to verify our theory. First, we take $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$. Since we have five modes, $I = 1, 2, 3, 4, 5$. For each mode $i \in I$ the control set $U_i = \{0, 1\} \times \{0, 1\}$. We solve the following nonlinear programming problem.

$$\begin{array}{rl} \text{maximize} & z_i \text{ with } i \in I \\ \text{subject to} & (\forall u \in U_i) \, ||f_i(x, u) - x_s||^2 \geq z^2 \\ & (\forall u \in U_i) \, ||f_i(x, u) - x_s||^2 \geq ||x - x_s||^2 \\ & (\forall j : 1 \leq j \leq m_i) \, 0 \geq \phi_{ij}(x) \\ & z_i \geq 0 \\ \text{final objective} & r^* := \max_{i \in J} z_i \end{array}$$

The definition of each $\phi_{ij}(x)$ for $i \in I$ is defined according to different mode. For example, in mode 1 we have

$$\begin{array}{rcl} x_1 & \geq & 0 \\ x_1 & \leq & 0.3 \\ x_2 & \geq & 0 \\ x_2 & \leq & 0 \end{array}$$

and in mode 2 we have

$$\begin{aligned} x_1 &\geq 0 \\ x_1 &\leq 0.3 \\ x_2 &\geq 0.3 \\ x_2 &\leq 1.0 \end{aligned}$$

By analyzing the dynamics of the two-tank system, it turns out in modes 1, 2 and 4 each state can be moved closer to the desired state by using the given control policy, namely for $i = 1, 2, 4$ the maximization above has no admissible solutions. In mode 3 the maximization yields $z_3 = 0.0253$, and in mode 5 we get $z_5 = 0.0138$. Therefore we have $r^* = \max\{0.0253, 0.0138\} = 0.0253$. Figure 5 depicts the simulation of the two-tank system under the given control policy with the initial state $x_0 = (0.4, 0.1)$, where the left picture depicts x_1, x_2 versus the time t, and the right picture depicts x_2 versus x_1, from

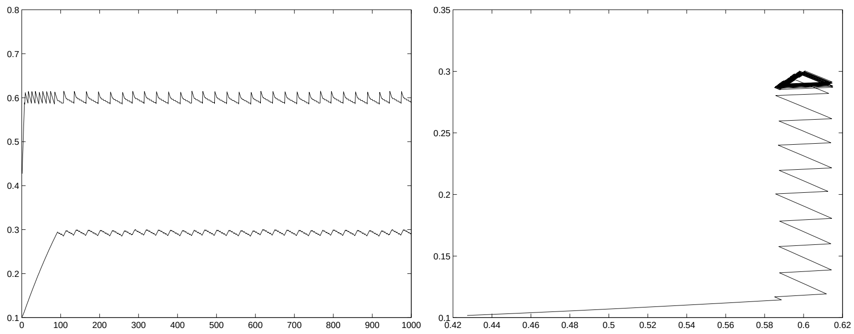

Figure 5: Case 1: $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$

which we can see the trajectory in this single instance is bounded within the range $[0.585, 0.615] \times [0.30, 0.285]$ after a finite number of time units. The maximum distance of a state in that range to x_s is $||((0.585, 0.285) - (0.60, 0.30)|| \approx 0.0212 < r^*$. It turns out that r^* is reached if we pick the initial state $x_0 = (0.5878, 0.2779)$ or the trajectory passes that state. Clearly the containable region $B(r^*, x_s)$ is not entirely within the desired set D.

In the second test we set $a_1 = 0.006$, $a_2 = 0.0002$ and $b = 0.0010$. The resulting radius is $r^* = 0.0051$. Figure 6 depicts the simulation of the two-tank system under the given control policy with the initial state $x_0 = (0.4, 0.1)$, where the left picture depicts x_1, x_2 versus the time t, and the right picture depicts

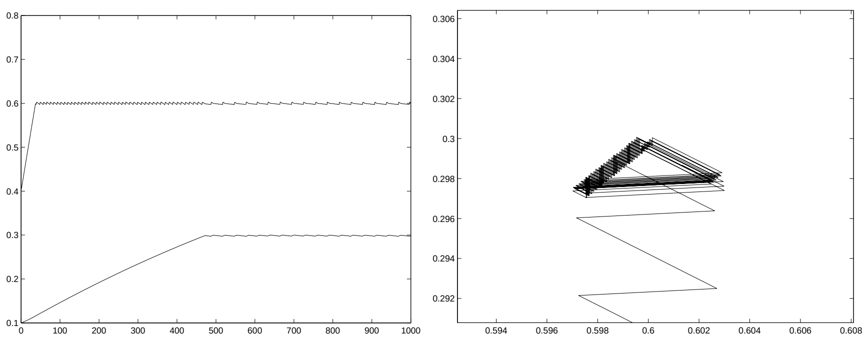

Figure 6: Case 2: $a_1 = 0.006$, $a_2 = 0.0002$ and $b = 0.001$

x_2 versus x_1. The trajectory in the right picture is bounded within the range $[0.597, 0.603] \times [0.30, 0.297]$ after a finite number of time units. The maximum distance of a state in that range to the desired state

x_s is $||((0.597, 0.297) - (0.60, 0.30)|| \approx 0.0042 < r^*$. It turns out that r^* is reached if we pick the initial state $x_0 = (0.5795, 0.2955)$ or the trajectory passes that state. Clearly this time the containable region $B(r^*, x_s)$ is entirely within the desired region D. At this point we need to check whether $B(r^*, x_s)$ is finitely reachable. First, we check whether or not $\partial B(r^*, x_s) \subseteq Q$, i.e. to compute the radius of the minimum closed ball $B(\hat{r}, x_s)$ that contains $\overline{Q}$. To this end, we solve the following nonlinear programming.

$$\begin{array}{rl} \text{maximize} & ||x^i|| \text{ with } i \in J \\ \text{subject to} & (\forall u \in U_i)\, ||f_i(x^i, u)||^2 \geq ||x^i||^2 \\ & (\forall j : 1 \leq j \leq m_i)\, 0 \geq \phi_{ij}(x^i) \\ \text{final objective} & \hat{r} := \max\limits_{i \in J} ||x^i|| \end{array}$$

It runs out that, in this case $\hat{r} = 0.0051$, which is equal to r^*. So $\partial B(r^*, x_s) \nsubseteq Q$, but for any $\epsilon > 0$ we have $\partial B(r^* + \epsilon, x_s) \subseteq Q$. Since $B(r^*, x_s) \subset B(r^* + \epsilon, x_s)$ we know that $B(r^* + \epsilon, x_s)$ is also a containable region. Clearly if we pick ϵ small enough, e.g. set $\epsilon = 0.0001$, then $B(r^* + \epsilon, x_s) \in D \cap Q$. Furthermore, we use the proposed nonlinear programming procedure to check whether the two-tank system is $(r^* + \epsilon)$-boundary active, and it turns out that

$$\left\{ \min_{x \in \partial X_i, ||x|| \geq r^* + \epsilon} \max_{u \in U_i} ||x|| - ||f_i(x, u)|| \middle| i \in I \right\} = \left\{ 0.0041, 0.0025, 0.0006, 0.0002, 0.0001 \right\}$$

Therefore, indeed the two-tank system is $(r^* + \epsilon)$-boundary active. By Proposition 3.2 the containable region $B(r^*+\epsilon, x_s)$ is finitely reachable. So finally we conclude that if we pick $a_1 = 0.006$, $a_2 = 0.0002$ and $b = 0.0010$, then the desired region D contains a finitely reachable containable region. Our simulations, although not exhaustive, show that the system's trajectory moves to the containable region within a finite number of time units.

4 Containable Region in Parameter-Uncertain Systems

In practical applications, due to the existence of external noise, the exact value of parameters in the target system may not be known, except for a bounded domain for each parameter. In a multi-mode system, for each mode X_i we have the following dynamics

$$x(k+1) = f_i(x(k), u(k), a_i(k)) \tag{9}$$

where $a_i(k) = [a_i^1(k), \cdots, a_i^p(k)]$ is the parameter vector of mode X_i at time instant k with each parameter$a_i^j(k)$ $(1 \leq j \leq q_i)$ belonging to a connected bounded domain $D_i^j = [b_i^j, c_i^j] \subset R$. But we don't know exact value of $a_i^j(k)$ at each time instance k. Suppose f_i is continuous over $\hat{X}_i \times D_i$ when u is fixed. Let $D_i := \prod_{j=1}^{q_i} D_i^j$. The control policy u^* at each state x is defined as follows:

$$u^* \in \arg \min_{u \in U_{\phi(x)}} \left[\max_{a_{\phi(x)} \in D_{\phi(x)}} ||f_{\phi(x)}(x, u, a_{\phi(x)})|| \right]$$

A set $S \subseteq R^n$ is a *containable region with respect to the parameter-uncertain model (9)* if

$$(\forall x \in S)\, f_{\phi(x)}(x, u^*, a_{\phi(x)}) \in S$$

For each $i \in I$ and $u \in U_i$ let

$$W_u := \left\{ x \in X_i \middle| \max_{a_i \in D_i} ||f_i(x, u, a_i)|| < ||x|| \right\}$$

and

$$Q_i := \bigcup_{u \in U_i} W_u$$

So Q_i is the set of all states in X_i where there is a control action $u \in U_i$ that can bring the system's trajectory closer to the desired state - the origin, no matter what values of those parameters are. Let

$$Q := \bigcup_{i \in I} Q_i$$

As before, $\overline{Q} := R^n - Q$. Let $\hat{X}_i := X_i \cup \partial X_i$ be the closure of X_i, where ∂X_i is the boundary.

$$r^* := \max_{i \in I} \max_{x \in \overline{Q} \cap \hat{X}_i} \min_{u \in U_i} \max_{a_i \in D_i} ||f_i(x, u, a_i)|| \tag{10}$$

Proposition 5 The closed ball $B(r^*)$ with r^* computed in (10) is a containable region.

Proof: For each $x \in B(r^*)$, suppose $i = \phi(x)$. If $x \in Q_i$ then by definition of Q_i we have

$$||f_i(x, u^*, a_i)|| < ||x|| \leq r^*$$

If $x \notin Q_i$ then $x \in \overline{Q} \cap X_i$. By (10) we have

$$\begin{aligned} ||f_i(x, u^*, a_i)|| &\leq \min_{u' \in U_i} \max_{a'_i \in D_i} ||f_i(x, u', a'_i)|| \\ &\leq \max_{x' \in \overline{Q} \cap \hat{X}_i} \min_{u' \in U_i} \max_{a'_i \in D_i} ||f_i(x', u', a'_i)|| \\ &\leq r^* \end{aligned}$$

In both cases we have $f_i(x, u^*, a_i) \in B(r^*)$, as required. ■

Nevertheless, to compute r^* is rather complicated. To make it computationally simple, we compute an alternative value

$$r_* := \max_{i \in I} \min_{u \in U_i} \max_{x \in \overline{Q} \cap \hat{X}_i} \max_{a_i \in D_i} ||f_i(x, u, a_i)|| \tag{11}$$

It is easy to see that $r^* \leq r_*$. Computing r_* in (11) can be converted into the following nonlinear programming problem.

$$\begin{aligned} \text{maximize} \quad & J_{i,u} := ||f_i(x, u, a_i)||^2 \text{ with } u \in U_i \\ \text{subject to} \quad & (\forall u' \in U_i)\, ||x||^2 \leq ||f_i(x, u', a_i)||^2 \\ & (\forall j : 1 \leq j \leq m_i)\, 0 \geq \phi_{ij}(x) \\ & (\forall j : 1 \leq j \leq q_i)\, b_i^j \leq a_i^j \leq c_i^j \\ \text{Set} \quad & J_i := \min_{u \in U_i} \sqrt{J_{i,u}} \\ \text{final objective} \quad & r_* := \max_{i \in I} J_i \end{aligned}$$

Notice that here we assume for each mode X_i, $\overline{Q} \cap \hat{X}_i \neq \varnothing$. Otherwise, we first need to check this condition for each mode either directly from the system description or by solving the following optimization problem

$$\begin{aligned} \text{minimize} \quad & L_i := ||x - y||^2 \\ \text{subject to} \quad & (\forall j : 1 \leq j \leq m_i)\, \phi_{ij}(x) \leq 0 \\ & (\forall u \in U_i)\, ||y||^2 \leq ||f_i(y, u, a_i)||^2 \end{aligned}$$

If $L_i = 0$ then the condition $\overline{Q} \cap \hat{X}_i \neq \varnothing$ holds. So the optimization for J_i can be performed.

To achieve a finite reachable containable region, we follow a similar procedure as described in previous sections. For each $i \in I$ let

$$g_i : \hat{X}_i \to R : x \mapsto g_i(x) := ||x|| - ||f_i(x, u^*, a_i^*)||$$

where a_i^* is the parameter vector when u^* is achieved. We have a similar result as Lemma 2.1.

Lemma 2 g_i is continuous.

Proof: By definition of g_i we have

$$g_i(x) = \|x\| - \min_{u \in U_i} \max_{a_i \in D_i} \|f(x, u, a_i)\|)$$

Thus, for each pair $x, x' \in \hat{X}_i$ we have

$$\begin{aligned}
|g(x) - g(x')| &= \left|(||x|| - \min_{u} \max_{a_i} ||f_i(x, u, a_i)||) - (||x'|| - \min_{u'} \max_{a'_i} ||f_i(x', u', a'_i)||)\right| \\
&= \left|\max_{u}(||x|| - \max_{a_i} ||f_i(x, u, a_i)||) - \max_{u'}(||x'|| - \max_{a'_i} ||f_i(x', u', a'_i)||)\right| \\
&\leq \max_{u} \left|(||x|| - ||x'||) - (\max_{a_i} ||f_i(x, u, a_i)|| - \max_{a'_i} ||f_i(x', u, a'_i)||)\right| \\
&\leq ||x - x'|| + \max_{u} \max_{a_i} ||f_i(x, u, a_i) - f_i(x', u, a_i)||
\end{aligned}$$

Since f_i is continuous on $\hat{X}_i \times D_i$ when u is given, for each $\epsilon > 0$, there exists δ_u such that

$$||(x, a_i) - (x', a'_i)|| < \delta_u \Rightarrow ||f_i(x, u, a_i) - f_i(x', u, a'_i)|| < \epsilon$$

Particularly choose $a_i = a'_i$. Then we have

$$\begin{aligned}
||x - x'|| < \delta_u &\Rightarrow (\forall a_i \in D_i)\, ||(x, a_i) - (x', a_i)|| < \delta_u \\
&\Rightarrow (\forall a_i \in D_i)\, ||f_i(x, u, a_i) - f_i(x', u, a_i)|| < \epsilon \\
&\Rightarrow \max_{a_i} ||f_i(x, u, a_i) - f_i(x', u, a_i)|| < \epsilon
\end{aligned}$$

Since U_i is finite, clearly we have

$$(\forall \epsilon > 0)(\exists \delta > 0)\, ||x - x'|| < \delta \Rightarrow ||x - x'|| + \max_{u} \max_{a_i} ||f_i(x, u, a_i) - f_i(x', u, a_i)|| < \epsilon$$

which means

$$(\forall \epsilon > 0)(\exists \delta > 0)\, ||x - x'|| < \delta \Rightarrow |g_i(x) - g_i(x')| < \epsilon$$

Thus, g_i is continuous, as required. ■

Here we use the same definition of *r-boundary activity* as defined in Def. 3.1. To verify it we can use a similar procedure as introduced in Section III, which strictly follows Def. 3.1:

$$(\forall i \in I)\, J_i := \min_{x \in \partial X_i,\, ||x|| \geq r} (||x|| - \min_{u \in U_i} \max_{a_i \in D_i} ||f_i(x, u, a_i)||) = \min_{x \in \partial X_i,\, ||x|| \geq r} \max_{u \in U_i} \min_{a_i \in D_i} (||x|| - ||f_i(x, u, a_i)||)$$

If $J_i > 0$ for each $i \in I$, then the system is r-boundary active. Nevertheless, computing J_i is not simple. So instead of computing J_i we compute

$$J_i^* := \max_{u \in U_i} \min_{x \in \partial X_i,\, ||x|| \geq r} \min_{a_i \in D_i} (||x|| - ||f_i(x, u, a_i)||)$$

Clearly, $J_i \geq J_i^*$. So if $J_i^* > 0$ then so is J_i. J_i^* can be computed as follows:

$$\begin{aligned}
\text{minimize} \quad & J_{i,u} := ||x|| - ||f_i(x, u, a_i)|| \text{ with } u \in U_i \\
\text{subject to} \quad & ||x|| \geq r \\
& \phi_{ij} = 0 \\
& (\forall p : 1 \leq p \leq m_i, p \neq j)\, \phi_{ip}(x) \leq 0 \\
& (\forall j : 1 \leq j \leq q_i)\, b_i^j \leq a_i^j \leq c_i^j \\
\text{objective} \quad & J_i^* = \max_{u \in U_i} J_{i,u}
\end{aligned}$$

If $J_i^* > 0$ for each $i \in I$, then the system is r-boundary active.

Proposition 6 Suppose r_* is computed from (11), and the system is r_*-boundary active. If $\partial B(r_*) \in Q$ then $B(r_*)$ is finitely reachable under the given control policy.

Proof: We follow a similar procedure as in Prop. 3.2 to show that

$$(\forall x_0 \in R^n)(\exists N(x_0, r_*) \in \mathbb{N})\, x(N(x_0, r_*)) \in B(r_*)$$

Suppose otherwise. Then there exists an initial state x_0 such that no such $N(x_0, r_*)$ exists. By Prop. 5.1, clearly $x_0 \notin B(r_*)$. Furthermore

$$(\forall k \in \mathbb{N})\, \|x(k)\| > r_*$$

By definition of r_* and the assumption $\partial B(r_*) \subseteq Q$, we have

$$(\forall k \in \mathbb{N})\, x(k) \in Q$$

Since $Q = \cup_{j \in I} Q_j$ and I is finite, there is an $i \in I$ such that X_i contains an infinite number of states along the trajectory. Let $\Delta_i = \{(x(k_l), a_i(k_l)) | l \in \mathbb{N}\} \subseteq \{(x(k), a_i(k)) | k \in \mathbb{N}\}$. Let

$$h_i : \hat{X}_i \times D_i \to R : (x, a_i) \mapsto h_i(x, a_i) := ||x|| - ||f(x, u^*, a_i)||$$

By definition of Q_i, for each $l \in \mathbb{N}$ we have

$$h_i(x(k_l), a_i(k_l)) := \|x(k_l)\| - \|f_i(x(k_l), u(k_l)^*, a_i(k_l))\| = \|x(k_l)\| - \|x(k_l + 1)\| > 0$$

Since

$$\sum_{(x, a_i) \in \Delta_i} h_i(x, a_i) \le \sum_{k=0}^{+\infty} ||x(k)|| - ||x(k+1)|| \le \|x_0\| - r_*$$

by Bolzano-Weierstrass Theorem, there exists a convergence sequence

$$\{h_i(x(j_l), a_i(j_l)) | (x(j_l), a_i(j_l)) \in \Delta_i \,\&\, l \in \mathbb{N}\} \subseteq h_i(((B(\|x_0\|) - B(r_*)) \cap (X_i \cup \partial X_i)) \times D_i)$$

such that

$$\lim_{l \to +\infty} h_i(x(j_l), a_i(j_l)) = 0$$

Clearly, $h_i(x(j_l), a_i(j_l)) \ge g_i(x(j_l))$ and by definition of Q we have $g_i(x(j_l)) \ge 0$. Thus,

$$\lim_{l \to +\infty} h_i(x(j_l), a_i(j_l)) = 0 \Rightarrow \lim_{l \to +\infty} g_i(x(j_l)) = 0$$

By Lemma 5.1 we know g_i is continuous on $X_i \cup \partial X_i$, and the set

$$\Omega_i := (B(\|x_0\|) - (B(r^*) - \partial B(r_*))) \cap (X_i \cup \partial X_i)$$

is compact, we have that

$$\lim_{l \to +\infty} g_i(x(j_l)) = 0 \in g_i(\Omega_i)$$

namely there exists $x \in \Omega_i$ such that $g_i(x) = 0$. But by definition of Q_i and the r_*-boundary activity of the system, we have

$$(\forall x' \in \Omega_i)\, g_i(x') > 0$$

which contradicts $g_i(x) = 0$. Therefore, $h_i(x(j_l), a_i(j_l))$ cannot have zero as the limit. ■

As an illustration we revisit that two-tank system (Figure 1). The system dynamics is the same as described in Section III and the desired region D is also the same. In case 1 we assume the system's nominal parameters are: $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$. But due to valve defects, the actual values of a_1 and b are not known exactly. We only know that $a_1 \in [0.027, 0.033] \subset R$ and $b \in [0.0045, 0.0055] \subset R$. The only online one-step lookahead control policy considers the worst case that a_1 and b can be any value within their individual domains. By (11) we can get that $r_* = 0.0355$. Figure 7 depicts the

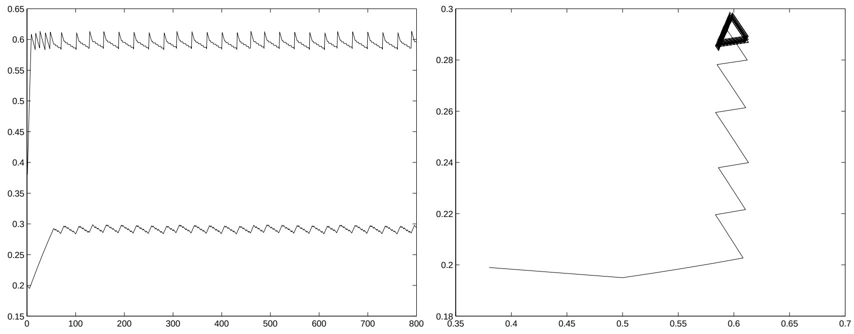

Figure 7: Case 1: $a_1 = 0.03$, $a_2 = 0.001$ and $b = 0.005$

simulation of system's dynamic behavior, where the initial state is $x_0 = (0.35, 0.20)$. The left picture shows $x_1(t)$ and $x_2(t)$ over t. The right picture shows $x_1(t)$ over $x_2(t)$, which indicates that the size of the containable region is a bit of smaller than r_* but is certainly out of the desired region D. Clearly, the size of the containable region with uncertain parameters is larger than the one with certain parameters illustrated in Figure 5.

In case 2 we assume the system's nominal parameters are: $a_1 = 0.006$, $a_2 = 0.0002$ and $b = 0.001$. Again, due to valve defects, the actual values of a_1 and b are not known exactly. We only know that $a_1 \in [0.0054, 0.0066] \subset R$ and $b \in [0.0009, 0.0011] \subset R$. The online one-step lookahead control policy considers the worst case that a_1 and b can be any value within their individual domains. By (11) we can get that $r_* = 0.0071$. Figure 8 depicts the simulation of system's dynamic behavior, where the initial

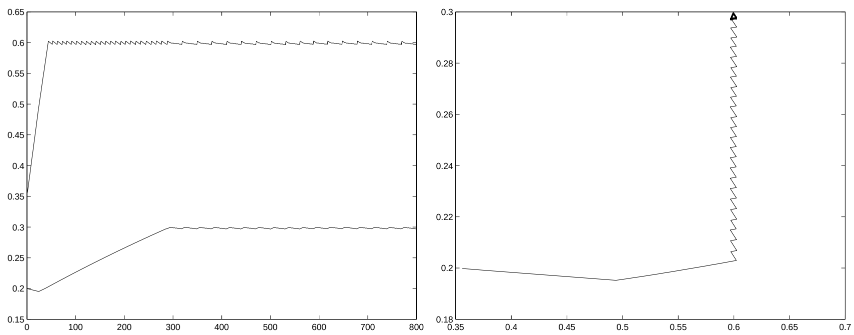

Figure 8: Case 1: $a_1 = 0.006$, $a_2 = 0.0002$ and $b = 0.001$

state is $x_0 = (0.35, 0.20)$. The left picture shows $x_1(t)$ and $x_2(t)$ over t. The right picture shows $x_1(t)$ over $x_2(t)$, which indicates that the size of the containable region is certainly within the desired region D. Again we can see that the size of the containable region with uncertain parameters is larger than the one with certain parameters illustrated in Figure 6.

5 Remarks

Although traditional discrete-time Lyapunov approaches targets a general class of dynamic systems, there is no general way to find an appropriate Lyapunov function for a given system. On the contrast, the proposed approach in this paper searches for a containable region, namely to compute $B(r^*)$, and decide the finite reachability of a containable region by merely using nonlinear programming. One issue concerning the proposed approach is that nonlinear programming may not always achieve the globally optimal solution, which may make $B(r^*)$ smaller than the true containable region. However, by choosing sufficient number of seeds during nonlinear programming and solving the dual problem to achieve the upper bound of the primal optimal solution, we can increase confidence in the correctness of r^* as the globally optimal solution. Another issue arises from the fact that a containable region is defined as a closed ball, which in reality may be unnecessarily large when $\overline{Q}$ is "long" and "narrow" in a geometric sense. But our experience on several practical systems indicates that the resulting estimates are quite helpful as a guidance to evaluate the system's performance. For example, in that two-tank system, the results indicate that small values of a_1, a_2 and b leads to a substantially small containable region. So we should increase the sampling rate to achieve sufficiently small values of those parameters. The analysis also indicates that in order to achieve fast transient behavior, we can deliberately introduce a set of large values of a_1, a_2 and b so that the system's trajectory can move quickly to the neighborhood of $\overline{Q}$, then those parameters are switched to small values so that a trajectory can move into a sufficiently small containable region. All these results are quite useful for a system designer to choose a system configuration, e.g. the values of parameters and the control set U.

Notice that in this paper we only consider a one-step lookahead control policy. The reason is that, as far as the same concept of containable region (i.e. expression (3)) is under consideration, a multi-step lookahead control policy is no better than the one-step lookahead policy in the sense that the former one will not lead to a smaller containable region than the latter one does. To see this, suppose at state $x \in R^n$ the l-step lookahead control policy yields a control action $\hat{u}$ at x, and the minimum radius of the containable region for the l-step lookahead policy is d. Then

$$r^* = \max_{x' \in \overline{Q}} ||f(x', u^*)|| = \max_{x' \in \overline{Q}} \min_{u \in U} ||f(x', u)|| \leq \max_{x' \in \overline{Q}} ||f(x', \hat{u})|| \leq d$$

Therefore, a one-step lookahead policy is the better choice for obtaining a good containable region than a multi-step lookahead policy.

6 Conclusions

In this paper a novel computational procedure was proposed to determine whether there is a finitely reachable containable set within a desired region, which is essentially a practical stability problem consisting of a containability problem and an attraction problem. We first explore the stability issue in a single-mode system, then extend it to a general multi-mode system.

In future work we will focus on developing efficient algorithm based on the proposed approach for specific classes of non-linear systems, and provide a less restrictive geometrical form, e.g. ellipsoid, for a containable region instead of a closed ball.

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